CENTRAL FUNCTIONS
AND THEIR PHYSICAL IMPLICATIONS

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Abstract

I define central functions $c(g)$ and $c'(g)$ in quantum field theory. These quantities are useful to study the flow of the numbers of vector, spinor and scalar degrees of freedom from the UV limit to the IR limit and justify the notions of secondary central charges recently introduced at criticality. Moreover, they are the basic ingredients for a description of quantum field theory as an interpolating theory between pairs of four dimensional conformal field theories. The correlator of four stress-energy tensors plays a key role in this respect. I analyse the behaviours of the central functions in QCD, computing their slopes in the UV critical point. To two-loops, $c(g)$ and $c'(g)$ point towards the expected IR directions. As a possible physical application, I argue that a closer study of the central functions might allow us to lower the upper bound on the number of generations to the observed value. A similar analysis is carried out in QED. Finally, candidate all-order expressions for the central functions are compared with the predictions of electric-magnetic duality.

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The fundamental properties of conformal field theories in four dimensions (CFT$_4$) are encoded into certain *central charges* c, c' and anomalous dimensions h. The primary central charge c is the Weyl squared contribution to the trace anomaly. Secondary central charges c' were defined in ref. [1], motivated by the observation that the OPE of the stress-energy tensor with itself does not close, in general [2]. New operators Σ are involved, h being their anomalous dimensions. The first radiative corrections to c and c' were computed in ref. [3] in the context of supersymmetric theories. The result suggests that the central charges are invariant under continuous deformations of CFT$_4$'s. Computations of h can be found both in [1] and [3]. Moreover, in [3] it was noted that to the second loop order in perturbation theory, it is possible, with no additional effort from the computational point of view, to extend the analysis off-criticality and study the first terms of quantities that what we can call *central functions*, which are the main subject of the present paper.

In general, conformal field theories are physically useful to describe the UV and IR limits of a quantum field theory (when they are well-defined). The purpose of the research pursued here is to identify and analyse the basic quantities that can give us a description of ordinary quantum field theories as theories interpolating between pairs of conformal field theories. The construction of natural and, to some extent, unique functions $c(g)$ and $c'(g)$[1] that interpolate between the critical values of the central charges is nontrivial. Their existence will be proved by a detailed renormalization group analysis of the correlator of four stress-energy tensors. In ref.s [1, 3] this issue was not considered and the present paper is mainly devoted to fill this gap. The long-range program is to study the flow of the central functions and, hopefully, extracting some nontrivial physical information. Precisely, the central functions $c(g)$ and $c'(g)$ allow one to study the flow of the numbers of vector, spinor and scalar degrees of freedom from the UV conformal fixed point to the IR conformal fixed point of a generic quantum field theory.

We recall from the discussion of ref. [1] that there are three fundamental central charges in CFT$_4$ (and, correspondingly, three central functions off-criticality). One considers the OPE of two stress-tensors and observes that only the three operators 1, $\bar{\varphi}\varphi$ and J^5_μ appear before $T_{\mu\nu}$ itself. The primary central charge c is associated with the identity operator and the two secondary charges c' are associated with the mass operator $\bar{\varphi}\varphi$ of the scalar fields and the axial current J^5_μ of the fermions. These charges are clearly independent and are those that we call "fundamental". At the moment, we cannot claim that they characterize CFT$_4$ completely. This issue is an open problem. Other independent charges could be generated by the construction that will be presented, but note that the next term of the TT-OPE is the stress-tensor itself, which gives back the primary central charge c.

The primary central function $c(g)$ is related to the two-point correlator $< T_{\mu\nu}(x)\, T_{\rho\sigma}(y) >$, while each one of the secondary central functions $c'(g)$ is related to a channel of the four-point correlator $< T_{\mu\nu}(x)\, T_{\rho\sigma}(y)\, T_{\alpha\beta}(z)\, T_{\gamma\delta}(w) >$. The reason why the correlators of the stress-energy tensor are relevant is that in counting the various kinds of degrees of freedom it is physically meaningful to use (external) gravity as a probe. It turns out that it is precisely the correlator

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[1]Here g denotes generically the set of coupling constants of the theory.

of four stress tensors that contains the relevant information for a correct description of CFT_4's and quantum field theories interpolating among them.

The four dimensional conformal field theories that are most similar to the two-dimensional ones satisfy the condition $\Sigma = 0$ or, equivalently, $c' = 0$, in which case the OPE of the stress-energy tensor closes and it is possible to define primary fields in the usual way. Unfortunately, there is only one example of this kind: the free vector boson. In the realm of supersymmetry, on the other hand, the OPE's of the stress-energy superfield close only for the free vector multiplet. These two simple conformal theories are relevant for QCD and supersymmetric QCD. In the infrared limit only the Goldstone bosons survive, so c should flow to zero in absence of quarks. Thus, the function $c(g)$ might be monotonically decreasing from the ultraviolet in pure Yang-Mills theory. The two-loop radiative correction to the free value exhibits indeed a decreasing behaviour (while c' is identically zero in perturbation theory). The situation is more intrigueing in presence of quarks, as we shall see. The second part of the paper is devoted to the detailed analysis of the slopes of the central functions in the free critical points of QCD and QED.

In some special models it might be possible to work out all-order expressions of the central functions, on the same footing, for example, as the NSVZ beta-function [4] of supersymmetric QCD. In the final part of this paper some candidate all-order expressions in supersymmetric theories will be compared with the predictions of electric-magnetic duality [5].

The existence of a c-theorem [6] in two dimensions motivated the search for some analogous property in four dimensions [7, 8]. The assumption of a monotonic behaviour of the central functions would strengthen considerably the physical discussion that we are going to present here. Nevertheless, just proceeding by general physical arguments will suffice to provide us with an interesting picture.

Definition of the first central function.

The correlator of two stress-energy tensors can always be written in the form

$$< T_{\mu\nu}(x)\, T_{\rho\sigma}(0) > = -\frac{1}{48\pi^4} X_{\mu\nu\rho\sigma}\left(\frac{c(g(x))}{x^4}\right) + \pi_{\mu\nu}\pi_{\rho\sigma}\left(\frac{f(\ln x\mu, g(\mu))))}{x^4}\right). \qquad (1)$$

where $\pi_{\mu\nu} = (\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)$ and $X_{\mu\nu\rho\sigma} = 2\pi_{\mu\nu}\pi_{\rho\sigma} - 3(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho})$. μ is the subtraction point and $g(x)$ is the running coupling constant. Often, $g(\mu)$ will be simply written as g. $T_{\mu\nu}$ is normalized as $e^{-1}e_\mu^a\, \delta S/\delta e_a^\nu$, S being the action and e_μ^a the vielbein.

The argument why only two independent functions can appear in eq. (1) is the following. The most general expression for $< T_{\mu\nu}(x)\, T_{\rho\sigma}(0) >$ is the sum of five independent tensors constructed with x_α and $\delta_{\alpha\beta}$. The most general expression for $< \partial_\mu T_{\mu\nu}(x)\, T_{\rho\sigma}(0) >$ is the sum of three independent terms. So, only two independent functions survive the imposition of the conservation condition $\partial_\mu T_{\mu\nu} = 0$.

The function c depends on x only via the gauge coupling $g(x)$, this being a consequence of the Callan-Symanzik equations and the finitness of the stress-energy tensor. This property is crucial, since a central function should depend on the value of the coupling g at a *single* scale. Strictly speaking, in presence of scalar fields ϕ the stress-tensor is not finite, since the

conservation law $\partial_\mu T_{\mu\nu} = 0$ allows it to mix with the operator $\pi_{\mu\nu}\phi^2$. Nevertheless, one can immediately see that this mixing affects only the function f and not c. The relation between renormalized and bare stress-energy tensors is $T^R_{\mu\nu} = T^B_{\mu\nu} + A\pi_{\mu\nu}\phi^2_B$, A being some appropriate renormalization constant. The matrix of renormalization constants for the operators $(\mathcal{S}_1, \mathcal{S}_2) \equiv$ $(T_{\mu\nu}, \pi_{\mu\nu}\phi^2)$ is then $\begin{pmatrix} 1 & A \\ 0 & Z_{\phi^2} \end{pmatrix}$, Z_{ϕ^2} being the renormalization constant of the mass operator. The Callan-Symanzik equations have to be applied to the set of correlators $< \mathcal{S}_i(x)\,\mathcal{S}_j(0) >$. Clearly, the correlators involving $\pi_{\mu\nu}\phi^2$ have the same form as the second term on the right hand side of eq. (1) and so cannot affect the first term.

The Callan-Symanzik equations can be safely applied inside the forth-order differential operators $X_{\mu\nu\rho\sigma}$ and $\pi_{\mu\nu}\pi_{\rho\sigma}$. When crossing these operators one has to worry about possible ambiguities of the form $|x|^4 \mathcal{P}(x)$ in the functions c and f, $\mathcal{P}(x)$ being a polynomial of order less than four (further restricted by Lorentz invariance to $a + bx^2$). However, these expressions cannot be generated in ordinary quantum field theory, since c and f have a perturbative expansion in powers of $\ln x\mu$.

The decomposition (1) is particularly convenient, because the natural (but nonlocal) traceless stress energy tensor $\mathcal{T}_{\mu\nu}$ defined as

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{3}\frac{1}{\Box}\pi_{\mu\nu}T,$$

$(T = T_{\alpha\alpha})$ satisfies

$$< \mathcal{T}_{\mu\nu}(x)\,\mathcal{T}_{\rho\sigma}(0) >= -\frac{1}{48\pi^4}X_{\mu\nu\rho\sigma}\left(\frac{c(g(x))}{x^4}\right). \tag{2}$$

The function $c(g)$ appearing in (1) or (2) has the correct properties to be taken as the all-order definition of primary central function. $c(g)$ correctly interpolates between the values of the central charges at the critical points. Moreover, it coincides with the definition used in [3] to the second loop order in perturbation theory around a free conformal fixed point.

Regulating the correlator (1) at $x = 0$ and taking the derivative with respect to $\ln\mu$, one gets a very simple equation for the integrated trace anomaly, namely a result proportional to

$$\tilde{c}(g)\,X_{\mu\nu\rho\sigma}\left(\delta^{(4)}(x)\right) + \text{trace terms.}$$

The omitted trace terms can be nonlocal. With a simple scheme choice, $\tilde{c}(g)$ is the Borel transform of $c(g(x))$ calculated at a special point. To be precise, let us write $c(g(x)) = \sum_n c_n(g)\,t^n$, $t = \ln x\mu$. Note that $c(g) = c_0(g)$. μ-independence imposes $c'_n(g)\beta(g) + (n+1)\,c_{n+1}(g) = 0$. We can regulate according to a formula that generalizes the prescription of ref. [9]

$$\frac{t^n}{x^4} \to -\frac{n!}{2^{n+1}}\Box\left(\sum_{k=0}^{n}\frac{2^k t^{k+1}}{(k+1)!}\frac{1}{x^2}\right) + a_n\delta(x), \tag{3}$$

the constants a_n parametrizing the scheme choice. With $a_n = 0$ one finds $\tilde{c}(g) = \sum_n n!\,c_n(g)/2^n$. A different scheme choice just adds a term proportional to the beta-function, which is immaterial for the computation the IR limit of the central charge. For this purpose, $\tilde{c}(g)$ is as good as

$c(g)$, since the difference between the two is always proportional to β. For example, one finds $c(g) = \tilde{c}(g) + 1/2\,\beta(g)\,\mathrm{d}\tilde{c}(g)/\mathrm{d}g$ in the scheme $a_n = 0$. Off-criticality, only $c(g)$ is scheme independent, while $\tilde{c}(g)$ is not. At criticality $c(g) = \tilde{c}(g)$ and this is true also at two-loops around a free theory.

The existence of a relationship, at criticality, between the central charge c and an anomaly in external field supports the statement that c is invariant under continuous deformations of CFT$_4$'s. Indeed, an anomaly in external field is only one loop in a theory in which all internal anomalies vanish (all higher order rescattering graphs resum to zero). In ref. [3] this property was explicitly checked to two-loops, both for c and c', and it was conjectured that the secondary central charge c' should also be invariant under continuous deformations of CFT$_4$. The idea is that c' is a generalization of the concept of anomaly in external field. Making this statement rigorous is still an open problem, but the two-loop results of [3] are a strong support to it.

Note that $\mathcal{T}_{\mu\nu}$ does not feel the "improvement ambiguity" $A\pi_{\mu\nu}\phi^2$ typical of scalar fields ϕ. Thus, the correct general statement for the finiteness of the stress-energy tensor is $\mathcal{T}_{\mu\nu}^R = \mathcal{T}_{\mu\nu}^B$.

Tracing eq. (1), we get tre two-point trace correlator

$$< T(x)T(0) >= 9\square^2 \left(\frac{f(\ln x\mu, g)}{x^4} \right). \tag{4}$$

In two dimensions the denominator x^4 is absent, $X_{\mu\nu\rho\sigma} = 2\pi_{\mu\nu}\pi_{\rho\sigma} - (\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho})$ is identically zero and only the function f survives. Now, our considerations in higher dimension, in particular formula (4), exclude any role of f at criticality, where only c should survive, and this raises an apparent puzzle. The solution is that for a very particular choice, namely f proportional to $\ln x\mu$, $\pi_{\mu\nu}\pi_{\rho\sigma}f$ is also traceless [8]. $f = c\ln x\mu$ is the expression valid at criticality, and f is indeed a good function interpolating between the critical values of the ultraviolet and infrared central charges[2]. The c-theorem [6] relies on this coincidence of two dimensions.

Definition of the other central functions.

The secondary central functions c' are generated by the correlator $< \mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\rho\sigma}(y)\mathcal{T}_{\alpha\beta}(z)\mathcal{T}_{\gamma\delta}(w) >$. We first work out the precise definition of c' and then explain the relationship with the correlator of four stress tensors. The discussion of the previous paragraph was just a warm-up for the more involved construction to be presented now. In particular, we shall see that the primary central function is just a very special secondary central function.

Generically, we can associate a secondary central function $c'_{\mathcal{O}}(g)$ with any operator $\mathcal{O}$, the most relevant cases being $\mathcal{O} = J_\mu^5$ and $\mathcal{O} = \bar{\varphi}\varphi$. In the simplest situation, one can assume that $\mathcal{O}$ does not mix with other operators under renormalization. More generically, the definition

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[2]$f = c\ln x\mu$ cannot be written as a function of g, of course. This is not the effect of improvement terms in the stress-energy tensor (absent in two dimensions). Rather, one has to note that in two dimensions the Callan-Symanzik equations cannot be safely applied inside the forth-order differential operator $\pi_{\mu\nu}\pi_{\rho\sigma}$. One ambiguity, namely the constant polynomial, survives. As a consequence, the equation $\dot{f}(\lambda + t, g) = \dot{f}(t, g(\lambda))$ is satisfied by the derivative of f with respect to $t = \ln x\mu$ and not by f itself. The most general solution is $f(t, g) = \int_{t_0}^t u(g(t'))dt'$ for a certain $u(g)$.

that we are going to give applies to a *set* of operators $\{\mathcal{O}_i\}$ that is closed under renormalization mixing and irreducible in this sense. For the moment we assume that the $\{\mathcal{O}_i\}$ have the same canonical dimension d.

For example, in ref. [3] the operators $\{\mathcal{O}_i\}$ are the various Konishi superfields K_i associated with the different irreducible representations of the matter multiplets. The K_i mix at the quantum level and this prevents us from constructing a function c' for each of them: instead, the K_i's contribute to the *same* central function c', unless some flavour symmetry or similar property allows one to split the set of Konishi superfields into different irreducible subsets $\{K_i\}_1, \dots \{K_i\}_n$, each of which will have its own c'-function $c'_1, \dots c'_n$.

In this paragraph we assume that the $\mathcal{O}_i$ are scalar operators. This simplifies the formulæ considerably and is sufficient to illustrate how to prodeed in generality.

Let $Z_{ij}(\ln x^2\mu^2)$ denote the renormalization constants of the operators $\mathcal{O}_i$. The correlators

$$< \mathcal{O}_i(x)\mathcal{O}_j(0) >= \frac{1}{x^{2d}} Z_{ik}(\ln x\mu, g) A_{kl}(g(x)) Z_{jl}(\ln x\mu, g) \tag{5}$$

define functions $A_{ij}(g)$ that depend only on g. The proof of this fact is a straightforward application of the Callan-Symanzik equations. On the other hand, the OPE

$$\mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\rho\sigma}(0) \sim \frac{1}{144\pi^2} X_{\mu\nu\rho\sigma} \left(\frac{B_i(g(x)) Z_{ij}^{-1}(\ln x\mu, g)}{x^{4-d}} \right) \mathcal{O}_j(0) \tag{6}$$

defines other functions $B_i(g)$. We used directly $\mathcal{T}_{\mu\nu}$ in (6) rather than $T_{\mu\nu}$, but an analysis similar to the one carried out for the primary central function can be repeated starting from $T_{\mu\nu}$. We set

$$c'_{\mathcal{O}}(g) \equiv B_i(g) A_{ij}(g) B_j(g). \tag{7}$$

This function interpolates between the appropriate critical values and coincides with the function computed in ref. [3] up to the second loop order included.

We stress that formulæ (5), (6) and (7) have a nontrivial meaning also at criticality, if the matter fields are in a reducible representation of the gauge group. Then, $Z_{ij} = |x\mu|^{h_{ij}(g)}$, $h_{ij}(g)$ being a matrix that, in general, is not symmetric. It is simple to work out examples of this kind starting from the results of [3].

We note that c' is scheme independent, while the quantities $A_{ij}(g)$ and $B_i(g)$ do depend on the subtraction scheme. To be precise, compatibility with the Callan-Symanzik equations requires that a scheme change dictated by $\mathcal{O}_i \rightarrow H_{ij}(g)\mathcal{O}_j$ be expressed as $Z(\ln x\mu, g) \rightarrow H(g)Z(\ln x\mu, g)H^{-1}(g(x))$, $A(g) \rightarrow H(g)A(g)H^t(g)$, $B(g) \rightarrow B(g)H^{-1}(g)$. Consequently, c' is invariant, because the finite subtractions simplify in expression (7): $c'(g) \rightarrow c'(g)$. This observation generalizes a remark made in the context of the two-loop computation of ref. [3] and shows, to all orders in perturbation theory, off-criticality and in the case of nontrivial renormalization mixing between operators, that the function $c'(g)$ is a physically meaningful quantity.

It is straightforward to extend the definition of $c'(g)$ to the case of mixing among operators of different canonical dimensions. This case is not of primary interest for the applications and is left to the reader.

With a suitable combination of limiting procedures, it is possible to single out the secondary central functions inside the correlator $< \mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\rho\sigma}(y)\mathcal{T}_{\alpha\beta}(z)\mathcal{T}_{\gamma\delta}(w) >$. Roughly, one has to "open" the four point correlator and look deeply inside. One first takes the limit in which two distances, for example $\delta_1 \equiv |x - y|$ and $\delta_3 \equiv |z - w|$, are much smaller than the other one, $\Delta_2 \equiv |x - w|$. This procedure factorizes two differential operators $X_{\mu\nu\rho\sigma}$ and $X_{\alpha\beta\gamma\delta}$, acting on a certain expression $\mathcal{E}(\delta_1, \Delta_2, \delta_3) = \sum_{\mathcal{O}} \mathcal{E}_{\mathcal{O}}(\delta_1, \Delta_2, \delta_3)$, with canonical powers distributed among the distances δ_1, Δ_2 and δ_3 according to the canonical dimension of the intermediate channel $\mathcal{O}$. Precisely,

$$< \mathcal{T}_{\mu\nu}(x)\, \mathcal{T}_{\rho\sigma}(y)\, \mathcal{T}_{\alpha\beta}(z)\, \mathcal{T}_{\gamma\delta}(w) >|_{\delta_{1,3} \ll \Delta_2} \sim \frac{1}{(144\pi^2)^2} \sum_{\mathcal{O}} \times$$

$$\overrightarrow{X}^{(1)}_{\mu\nu\rho\sigma} \left(\frac{B_i(g(\delta_1))Z^{-1}_{ij}(\delta_1, g)}{\delta_1^{4-d}} \frac{Z_{jk}(\Delta_2, g)A_{kl}(g(\Delta_2))Z_{ml}(\Delta_2, g)}{\Delta_2^{2d}} \frac{Z^{-1}_{nm}(\delta_3, g)B_n(g(\delta_3))}{\delta_3^{4-d}} \right) \overleftarrow{X}^{(3)}_{\alpha\beta\gamma\delta} \quad (8)$$

Then, taking the limit $\delta_1 = \Delta_2 = \delta_3 = \lambda$, one gets $\mathcal{E}_{\mathcal{O}}(\lambda) = c'_{\mathcal{O}}(g(\lambda))/\lambda^8$.

In CFT$_2$, the only channels $\mathcal{O}$ are the stress tensor itself and its derivatives[3], so one gets the usual central charge. In higher dimensions, instead, one has many channels $\mathcal{O}$ and many corresponding central charges (or functions, off-criticality)[4]. Consequently, the basic correlator that one has to consider in order to properly describe higher dimensional CFT's or quantum field theories interpolating among them is precisely the correlator of four stress tensors rather than the correlator of two.

It is true in any dimension that the secondary central function associated with the channel $\mathcal{O} = \mathcal{T}_{\mu\nu}$ coincides with the primary central function itself: $c'_{\mathcal{T}_{\mu\nu}}(g) = c(g)$. This is because $B = 1$, $A(g) = c(g)$ in this very special case.

Before going on, let us summarize the idea behind our construction. At first, one might think that it is consistent to associate a central function to any operator $O(x)$, i.e. the coefficient of the identity operator in the OPE $O(x)O(0)$. However, this is not the case. First of all, one has to eliminate the Z-factors, otherwise one does not define just a function of $g(x)$, but a function of $g(x)$ and x separately: see, for example, eq. (5). But even after getting rid of the Z-factors, $A(g)$ turns out to be a scheme dependent quantity, as discussed below eq. (7). Moreover, due to operator mixing, the quantity $A_{ij}(g)$ is in general a matrix and not a function. Finding appropriate B_i's to paste to it while preserving a physical meaning is not trivial. For this reason, just those operators O_i that appear in the stress-tensor OPE are considered here and their appearance in the stress-tensor OPE automatically defines the appropriate B_i's. It is clear, then, that the relevant correlator is the one of four stress-tensors.

Finally, we note that these observations cannot be worked out while staying at criticality, yet they have relevant consequences on CFT$_4$: for example, they show that only those quantities that can be promoted to central functions off-criticality deserve to be called central charges at

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[3]The identity operator, on the other hand, cannot be considered as an intermediate channel, since it contributes only to disconnected diagrams.

[4]Nevertheless, how many of these functions are independent is still an open problem. At the moment we can just say that they are at least three.

criticality. It is thanks to the construction presented here that the notion of secondary central charges used for c' in [1, 3] is firmly justified.

When a quantum field theory admits other conserved currents J (for example flavour currents), the construction presented here can be generalized in an obvious way to the correlators $< JJ >$ and $< JJJJ >$. J replaces $\mathcal{T}$ and defines "flavour" primary and secondary central charges.

Having constructed natural central functions $c(g)$ and $c'(g)$ in a generic quantum field theory, we are now entitled to study their behaviours in perturbation theory. We have to expect that the central functions do not share most of the properties of the two dimensional c-function, apart from interpolating between the desired UV and IR values. For example, in two dimensions the c-function is stationary ($\mathrm{d}c/\mathrm{d}\alpha = 0$, $\alpha = g^2$) at criticality, as a consequence of the c-theorem. This property does not hold in four dimensions, neither for $c(g)$, nor for $c'(g)$, as the results of ref. [3] clearly show. So, the first interesting quantities to be computed are the slopes of the central functions at criticality. Were we able to compute the central functions of QCD to all orders, we would presumably be able to extract relevant properties of the infrared limit of the theory, for example if it confines or not. Moreover, we have given a rigorous justification of the off-critical analysis of [3].

Analysis of the slope of the first central function.

It is not clear whether in QCD the beta function tends to zero or minus infinity in the infrared limit. If there is no infrared fixed point, then it might be meaningless to count the degrees of freedom via the central functions. So, we assume that there is an IR fixed point. Moreover, since our analysis focusses on a neighborhood of the ultraviolet fixed point, we can neglect all the quark masses and work effectively with massless QCD.

Let c' refer to the fermions and c'' to the scalar fields. In pure Yang-Mills theory, we have $c_{UV} = \frac{1}{10}\mathrm{dim}G \equiv \frac{1}{10}N_v$ and $c'_{UV} = c''_{UV} = 0$. We expect $c_{IR} = c'_{IR} = c''_{IR} = 0$. We must note that c' and c'' are perturbatively zero. Non-perturbative effects like the glueballs, for example, disappear in the far infrared because they are massive, but at intermediate energies they are present. The situation is less simple in presence of quarks, where c''_{IR} should measure the number of Goldstone bosons, that are nonperturbative effects surviving in the infrared limit. With gauge group $SU(N_c)$ and N_q quarks in the fundamental representation, we have $c_{UV} = \frac{1}{10}N_v + \frac{1}{20}N_qN_c$, $c'_{UV} = N_qN_c$, $c''_{UV} = 0$ and we expect that in the infrared, where only the Goldstone bosons survive, $c_{IR} = \frac{1}{120}(N_q^2 - 1)$, $c'_{IR} = 0$ and $c''_{IR} = N_q^2 - 1$. In QCD we have $c_{UV} > c_{IR}$ and this inequality holds for a wide range of values of N_c and N_q. For example, for $N_c = 3$ any value N_q in the range of asymptotic freedom is allowed.

The two-loop computation gives

$$c\left(\alpha_s\right) = \frac{1}{20}\left[2N_v + N_qN_c - \frac{5}{9}N_v\frac{\alpha_s}{\pi}\left(2N_c - \frac{7}{8}N_q\right)\right] + \mathcal{O}(\alpha_s^2),$$

$$\left.\frac{\mathrm{d}c(\alpha_s)}{\mathrm{d}\alpha_s}\right|_{\alpha_s=0} = -\frac{N_v}{36\pi}\left(2N_c - \frac{7}{8}N_q\right).$$

This result can be read from [10] or derived by combining [11] with [12], after noting that the

matter contribution is the same in QED and QCD to the order considered (apart from the obvious change in the representations and their Casimirs).

This shows that in pure Yang-Mills theory the central function c does decrease, the two-loop correction to c having the same sign as the one-loop beta-function. The presence of matter, instead, produces the opposite effect, similarly to what happens for the beta function. However, the effect of matter on the c function is much stronger than the effect on the beta function, to the extent that an inequality like $dc(\alpha_s)/d\alpha_s|_{\alpha_s=0} < 0$ requires

$$N_q < \frac{16}{7} N_c. \tag{9}$$

Asymptotic freedom, instead, imposes the less restrictive condition $N_q < \frac{11}{2} N_c$. For $G = SU(3)$ (9) becomes noticeably $N_q \leq 6$. This could be just a coincidence or the sign of some physical principle, telling us, for example, that once the behaviour of the first radiative correction off criticality is "wrong", then the central function cannot tend to the appropriate infrared limit. Instead, $c(\alpha_s)$ cannot have a simple behaviour with $G = SU(3)$ and more than six quark and there will be a value of the coupling constant α_s for which $dc_s(\alpha_s)/d\alpha_s = 0$. Pursuing this kind of investigation further might allow us to lower the upper bound on the number $N_q/2$ of generations (fixed by asymptotic freedom to be $N_q/2 \leq 8$) to the observed value.

Let us repeat the analysis in QED. We have [11]

$$c_e(\alpha) = \frac{1}{20} \left(2 + N_f + \frac{35}{36} \frac{\alpha}{\pi} N_f \right) + \mathcal{O}(\alpha^2), \quad \frac{dc(\alpha)}{d\alpha}\bigg|_{\alpha=0} = \frac{7}{144} \frac{N_f}{\pi} > 0. \tag{10}$$

We see that now $c_e(\alpha)$ is greater than $c_{e\ free}$. Since the theory is infrared free, (10) shows that $c_e(\alpha)$ decreases towards the infrared. This behaviour agrees with the one noted in QCD (which was decreasing from the ultraviolet). There is no constraint, in QED, on the maximum number N_f of charged Dirac fermions, which, on the other hand, would sound much less natural than an upper bound on the number of generations.

Computation of the second central function to the second loop order in QCD and analysis of its slope.

To compute c', we use the following trick. A straightforward analysis of the relevant Feynman diagrams shows that c' can only have the form

$$c'(\alpha_s) = N_q \dim R + y\, N_q N_v\, C(R)\, \frac{\alpha_s}{\pi},$$

where R denotes the representation of the fermions and $C(R)$ is the corresponding Casimir. y is an unknown numerical coefficient that we have to determine. Now, in the particular case of one Majorana fermion ($N_q = 1/2$) in the adjoint representation, the theory is supersymmetric and $c' = N_v/2 + y(N_v N_c/2)\, \alpha_s/\pi$. We observe that in supersymmetric QCD, as proved in [1, 3], c and c' have to be proportional to each other, since a single superfield, $J_{\alpha\dot\alpha}$ contains both the stress-energy tensor and the R-current J_5. Since $c = N_v/8 - (N_v N_c/32)\, \alpha_s/\pi$ [3], we derive

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$y = -1/4$. We conclude that in QCD we have

$$c'(\alpha_s) = N_q N_c - \frac{1}{8} N_q N_v \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2), \quad \frac{\mathrm{d}c'(\alpha_s)}{\mathrm{d}\alpha_s}\bigg|_{\alpha_s=0} = -\frac{1}{8} \frac{N_q N_v}{\pi} < 0. \tag{11}$$

In QED this result translates into $c'_e(\alpha) = N_f - (N_f/4)\,\alpha/\pi + \mathcal{O}(\alpha^2)$.

We see from (11) that the effective number of fermions decreases from the ultraviolet, which is good because it should tend to zero in the far infrared.

We can consider also the quantity

$$N_v^{eff} \equiv 10c - \frac{c'}{2} = N_v - \frac{5}{18} \frac{\alpha}{\pi} N_v \left(2N_c - \frac{11}{10} N_q \right). \tag{12}$$

Since c counts the total number of degrees of freedom (with appropriate weights), while c' counts the number of fermions, the above difference should count the effective number of vectors. We see, however, that in QCD with 6 quarks the above function is slightly increasing from the ultraviolet, while we would expect it to decrease, since in the far infrared the vectors should disappear completely. $N_q = 6$ still plays a special role, but now 6 is the minimum number of quarks for the above function to *increase*. In general, we see that the functions c and N_v^{eff} are very close to be constant for $N_q = 2N_c$.

Candidate all-order expressions and electric-magnetic duality.

An interesting problem is to work out all-order expressions for the central functions. This might be very difficult in ordinary theories and could be easier in supersymmetric theories. In particular, let us consider the formulæ $c(g, Y) = (3N_v + N_\chi + N_v \beta_g/g - \gamma_i^i)/24$ and $c'(g, Y) = N_\chi + 2\gamma_i^i$ appearing in [3]. They hold to two-loops in the most general N=1 supersymmetric theory. One might ask oneself if these very simple formulæ are all-order valid[5]. We want now to compare some consequences of this hypothesis with the predictions of electric-magnetic duality [5].

We can write (the reader is referred to [3] for the notation),

$$D \equiv 24c + \frac{1}{2}c' = 3N_v + \frac{3}{2}N_\chi + N_v \frac{\beta_g(g, Y)}{g},$$

which implies

$$D_{UV} = D_{IR}. \tag{13}$$

For an electric theory with N_f quarks, $N_\chi = 2N_f N_c$. We shall compare the prediction (13) with the expectations of electric magnetic duality in the "conformal window" $3/2N_c \leq N_f < 3N_c$, where both the electric and magnetic theories are asymptotically free and we can compute D in the UV limits. For the electric theory we have $D_{el} = 3(N_c^2 - 1 + N_c N_f)$. For the magnetic theory we compute $D_{mag} = 3(N_c^2 - 1 - 3N_f N_c + 5N_f^2/2)$. The two values do not coincide in general. We conclude that the exactness of the expressions of c and c' of [3]

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[5] The expression of anomalous dimension h given in [3], on the other hand, cannot be expected to be all-order valid.

and electric-magnetic duality lead to physical predictions that are incompatible. Hopefully, electric-magnetic duality will teach us how to improve our two-loop formulæ.

Conclusions.

We have worked out all-order definitions of certain central functions that might be useful to extract relevant properties of quantum field theories. The construction is non-trivial, but it is also quite general, natural and, to some extent, unique. Not all the quantities of CFT_4 can be promoted to central functions off-criticality and we showed which ones can and how.

We do not have a method for deriving all-order expressions of the central functions, for the moment. This task might be very difficult, but the properties singled out in [3] and here make us hope that it is affordable. In supersymmetric theories, in particular, non-perturbative ideas might allow us to improve the two-loop expressions worked out in ref. [3].

Here we have explored various issues to the second loop order in perturbation theory. In particular, we computed the slopes of the central functions in the UV fixed point of massless QCD and in the IR fixed point of massless QED. We have pointed out some special role played by $N_q = 6$ for $N_c = 3$ in QCD and, as a possible physical application, we have stressed that a closer investigation of the central functions might lower the theoretical upper bound on the number of generations to the observed value.

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